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A VARIATIONAL APPROACH TO CERTAIN NONLINEAR BOUNDARY VALUE PROBLEMS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled A VARIATIONAL APPROACH TO CERTAIN NONLINEAR BOUNDARY VALUE PROBLEMS submitted by Elizabeth K. Leonard in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

We show the existence of a solution, positive and continuous on $0 \leq t < \infty$, and tending to 0 as $t \to \infty$, of the differential equation

$$\ddot{x} + 2t^{-1}\dot{x} - x + a(t)x^{k} = 0, \quad 1 < k < 4 . \tag{1}$$

This is a generalization of

$$\ddot{x} + 2t^{-1}\dot{x} - x + x^{2} = 0$$

which arose in connection with work by Takahashi concerning the structure of the nucleon core, and studied by J.L. Synge.

First we consider the simpler problem

$$\ddot{u} - u + a(t)t^{1-k}u^{k} = 0, \quad 1 < k < 4 \tag{2}$$

and set up a variational problem whose solution, if it exists, must satisfy (2), be continuous on $0 \leq t < \infty$, with $\lim_{t \to 0} t^{-1}u(t) < \infty$ and $\lim_{t \to \infty} t^{-1}u(t) = 0$. Then a direct method is used to establish the existence of a solution to the variational problem.

We next show that the variational method can be used to prove the existence of a solution to the boundary value problem

$$u'' + \lambda P(x)u^{\nu+1} = 0, \quad u(0) = u(1) = 0, \quad 0 < x < 1.$$

However, the procedure is simpler, as no singularities are present in the interval considered.

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CHAPTER I

PHYSICAL BACKGROUND AND PRELIMINARY REDUCTIONS

Takahashi studied a simple model for the structure of the nucleon core in [1]. His model was a system consisting of a non-relativistic nucleon and a neutral meson field. He obtained the following pair of differential equations for the unknowns ϕ and V:

$$\begin{aligned}(\Delta + \varepsilon)\phi(\underset{\sim}{x}) &= 2MfV(\underset{\sim}{x})\phi(\underset{\sim}{x}),\\ (\Delta - \mu^2)V(\underset{\sim}{x}) &= f\phi^*(\underset{\sim}{x})\phi(\underset{\sim}{x}), \quad x \in \mathbb{R}^3,\end{aligned} \tag{1}$$

subject to the normalization condition

$$\int \phi^*(\underset{\sim}{x})\phi(\underset{\sim}{x})d^3\underset{\sim}{x} = 1. \tag{2}$$

Here, Δ is the Laplacian operator, $\phi(x)$ the nucleon wave function, $V(x)$ the (real-valued) meson potential, f is the coupling constant between nucleon and meson field, while ε, M and μ are constants involving the mass and energy level of the system. Takahashi showed that a perturbation approach would not work, and he then applied a variational technique (the Ritz method).

In [2], Synge attacked this system of equations under the following special assumptions:

(i) $\varepsilon = -\mu^2$,

(ii) $\phi = kV$ where k is some complex constant,

(iii) solutions have spherical symmetry.

Because of (i) and (ii), the equations (1) become

$$(\Delta - \mu^2)V(\underset{\sim}{x}) = 2MfV^2(\underset{\sim}{x}),$$

$$(\Delta - \mu^2)V(\underset{\sim}{x}) = fk^*kV^2(\underset{\sim}{x}).$$

In order for these equations to be consistent $k^*k = 2M$; we have, then, the single partial differential equation

$$(\Delta - \mu^2)V(\underset{\sim}{x}) = 2MfV^2(\underset{\sim}{x}) \tag{3}$$

Letting $V = V(\underset{\sim}{x})$, $\mu\underset{\sim}{x} = \underset{\sim}{\bar{x}}$ and $W(\underset{\sim}{\bar{x}}) = V\,\frac{\underset{\sim}{\bar{x}}}{\mu}$ in equation (3), we have

$$\mu^2(\bar{\Delta} - 1)W = 2MfW^2,$$

that is,

$$(\bar{\Delta} - 1)W = \frac{2Mf}{\mu^2}\,W^2 .$$

Making a further substitution $\bar{V}(\underset{\sim}{\bar{x}}) = \frac{2Mf}{\mu^2}\,W(\underset{\sim}{\bar{x}})$, Synge obtained

$$(\bar{\Delta} - 1)\bar{V} = \bar{V}^2 .$$

Since $\Phi = kV$, the normalization condition (2) becomes $\int V^2(\underset{\sim}{x})d^3\underset{\sim}{x} = \frac{1}{k^*k} = \frac{1}{2M}$, and using the above transformation we obtain

$$\int W^2(\underset{\sim}{\bar{x}})d^3\underset{\sim}{\bar{x}} = \frac{\mu^3}{2M} .$$

Hence,

$$\int \bar{V}^2(\underset{\sim}{\bar{x}})d^3\underset{\sim}{\bar{x}} = \frac{\mu^3}{2M} \cdot \frac{4M^2f^2}{\mu^4} = \frac{2Mf^2}{\mu} . \tag{5}$$

Since he assumed the condition of spherical symmetry, letting

$r = \sqrt{\sum_{j=1}^{3} \bar{x}_j^2}$ and $g(r) = \bar{V}(\bar{x})$, so that

$$\frac{\partial \bar{V}}{\partial \bar{x}_j} = g'(r)\frac{\bar{x}_j}{r} \quad \text{and} \quad \frac{\partial^2 \bar{V}}{\partial \bar{x}_j^2} = \frac{g''(r)r - g'(r)}{r^2} \cdot \frac{\bar{x}_j^2}{r} + \frac{g'(r)}{r}, \quad j = 1,2,3,$$

he obtained from (4),

$$\frac{d^2\bar{V}}{dr^2} + 2r^{-1}\,\frac{d\bar{V}}{dr} - \bar{V} - \bar{V}^2 = 0,$$

that is,

$$\overline{V}'' + 2r^{-1}\overline{V}' - \overline{V} - \overline{V}^2 = 0, \tag{6}$$

and from (5) $\displaystyle\int_0^\infty r^2\overline{V}^2 dr = \frac{Mf^2}{2\pi\mu}$.

The problem was to find a solution $\overline{V}$ of (6) finite for $0 \le r < \infty$, with $\overline{V}(\infty) = 0$.

In order to do so, Synge wrote $\overline{V} = x(t)$, $r = t$, so that

$$x'' + 2t^{-1}x' - x - x^2 = 0, \quad 0 \le t < \infty, \tag{7}$$

with $x(t) \to 0$ as $t \to \infty$.

Equation (6) can be generalized to dimension N from 3 in which case the equation is

$$x'' + \alpha t^{-1}x' - x - x^2 = 0 \quad \text{where} \quad \alpha = N - 1.$$

The corresponding first-order system is

$$\begin{aligned} x' &= y \\ y' &= -\alpha t^{-1}y + x + x^2 . \end{aligned} \tag{8}$$

Any solution to (8) is regarded as the motion of a representative point with coordinates (x,y) in a phase-plane. Synge went on to define the "energy" E as

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 - \frac{1}{3}x^3$$

so that $E' = -\alpha t^{-1}y^2 \le 0$ and the "energy" decreases steadily, so the motion of the representative point is "downhill" across the level curves $E = \text{constant}$.

Since he was mainly concerned with the 3-dimensional case $(\alpha = 2)$, he then considered the equation (7) which is equivalent to the system (8) with $\alpha = 2$. In this case, on the basis of qualitative considera-

tions about the phase plane and detailed numerical calculations, Synge came to the conclusion that it appears that (7) has a unique solution such that $x(0)$ is finite and $x(t) \to 0$ as $t \to \infty$ where $x(0) = -4.19169$ (approximately).

In [3], Nehari settled the question of the existence of a positive solution of

$$\ddot{x} + 2t^{-1}\dot{x} - x + x^{k} = 0, \quad 1 < k < 4 \tag{9}$$

satisfying $\lim_{t \to \infty} x(t) = 0$ using a variational approach. However, he could not prove uniqueness. The extra range of powers represented by k allows a slightly more general relation between ϕ and V than (ii). In [4], Sansone gave an exhaustive treatment of this problem, using a wide variety of techniques from the qualitative theory of ordinary differential equations; in particular, he proved uniqueness. In this thesis, we treat a generalization of the equation (9), using Nehari's variational approach.

CHAPTER II

ANALYSIS OF THE MATHEMATICAL PROBLEM

1. We consider the question of the existence of a solution to the ordinary differential equation

$$\ddot{x} + \frac{2\dot{x}}{t} - x + a(t)x^k = 0 \quad \left(x = x(t)\right) \tag{1a}$$

satisfying the conditions:

$$\left.\begin{array}{ll} \text{(i)} & x(t) > 0 \quad \text{for} \quad 0 < t < \infty \\ \text{(ii)} & x(t) \text{ is continuous for } 0 \leq t < \infty \\ \text{(iii)} & \lim\limits_{t \to \infty} x(t) = 0, \end{array}\right\} \tag{1b}$$

where k is constant, $1 < k < 4$, and $0 \leq a(t) < M$ $\left(a(t) \not\equiv 0\right)$ is continuous on $[0,\infty]$, also $a(t_k) > 0$ for some $\{t_k\} \downarrow 0$.

We will show that (1a) has a solution satisfying (1b).

We shall follow the variational approach of Nehari [3].

2. <u>The Variational Problem</u>. Letting $u(t) = t\,x(t)$, we get $\dot{x} = \dot{u}t^{-1} - t^{-2}u$, $\ddot{x} = t^{-1}\ddot{u} - t^{-2}\dot{u} + 2t^{-3}u - t^{-2}\dot{u}$ and equation (1a) becomes

$$\ddot{u} - u + a(t)t^{1-k}u^k = 0. \tag{2}$$

We will show that (2) has a solution which satisfies:

(i) $u(t)$ is continuous for $0 \leq t < \infty$

(ii) $\lim\limits_{t \to 0} t^{-1}u(t) < \infty$

(iii) $\lim\limits_{t \to \infty} t^{-1}u(t) = 0$

by setting up a variational problem whose solution, if it exists, must satisfy (2) and the conditions (i)-(iii) above. We will use a direct method to establish the existence of the solution of the variational problem.

The problem is to minimize

$$J(y) \equiv \int_0^\infty (\dot{y}^2 + y^2)\,dt \tag{3}$$

where $y(t)$ ranges over the class A of functions which

(a) are non-negative for $0 \leq t < \infty$,

(b) are continuous for $0 \leq t < \infty$,

(c) have a piecewise continuous derivative for $0 \leq t < \infty$,

(d) vanish for $t = 0$,

(e) are normalized by the condition $\int_0^\infty a(t) \frac{y^{k+1}}{t^{k-1}}\,dt = 1$, (4)

(f) are such that the integral (3) exists.

The existence of (4) is implied by the existence of (3), so we need not assume it explicitly. To see this, we argue as follows. Since $y(0) = 0$,

$$y^2(t) = \left(\int_0^t \dot{y}\,d\tau\right)^2 \leq t \int_0^t \dot{y}^2 d\tau \leq t \int_0^\infty (\dot{y}^2 + y^2)\,d\tau = t\,J(y) \tag{5}$$

and

$$y^2(t) = 2\int_0^t y\,\dot{y}\,d\tau \leq \int_0^t (\dot{y}^2 + y^2)\,d\tau \equiv \gamma^2 \qquad (\gamma > 0). \tag{6}$$

Also for any function $y(t)$ with $\dot{y} \in L_2[0,\infty)$ and $y(0) = 0$, we have the following (by Littlewood's inequality which is proven in [5])

$$\int_0^T \frac{y^2}{t^2}\,dt \leq 4\int_0^T \dot{y}^2 dt.$$

Thus, if $T > 1$, we get, since $a(t)$ is bounded by M,

$$\int_0^T a(t)t^{1-k}y^{k+1}dt = \int_0^1 a(t)t^{1-k}y^{k+1}dt + \int_1^T a(t)t^{1-k}y^{k+1}dt$$

$$= \int_0^1 a(t)\,\frac{y^2}{t^2}\,t^{\frac{5-k}{2}}\,y^{k-1}\,t^{\frac{1-k}{2}}\,dt + \int_1^T a(t)y^2[t^{1-k}y^{k-1}]dt$$

$$\leq \gamma^{k-1}\int_0^1 a(t)\,\frac{y^2}{t^2}\,t^{\frac{5-k}{2}}\,dt + \gamma^{k-1}\int_1^T a(t)\,y^2 dt, \quad (\gamma = \gamma(T))$$

since

$$y^2(t) \leq t\gamma^2 \Rightarrow y^{k-1}\,t^{\frac{1-k}{2}} \leq \gamma^{k-1}, \quad 0 < t < 1;$$

and

$$t^{1-k}y^{k-1} \leq y^{k-1} \leq \gamma^{k-1}, \quad k > 1, \quad t > 1.$$

So

$$\int_0^T a(t)t^{1-k}y^{k+1}dt \leq M\gamma^{k-1}\int_0^1 \frac{y^2}{t^2}\,dt + M\gamma^{k-1}\int_1^T y^2 dt \qquad (\text{since } t^{\frac{5-k}{2}} \leq 1, \ 0 < t < 1)$$

$$\leq 4M\gamma^{k-1}\int_0^T \dot{y}^2 dt + M\gamma^{k-1}\gamma^2$$

$$\leq 5M\gamma^{k+1}.$$

Thus,

$$\int_0^T a(t)t^{1-k}y^{k+1}dt \leq 5M\gamma^{k+1}$$

and

$$\left(\int_0^T a(t)t^{1-k}y^{k+1}dt\right)^2 \leq 25M^2\gamma^{2(k+1)} = 25M^2\left(\int_0^T (\dot{y}^2+y^2)dt\right)^{k+1}.$$

That is,

$$\left(\int_0^T a(t)t^{1-k}y^{k+1}dt\right)^2 \leq 25M^2\left(\int_0^T (\dot{y}^2+y^2)dt\right)^{k+1}, \quad k \leq 5, \quad \forall\, T \in (0,\infty). \tag{7}$$

(7) seems to show that the equation (2) may have a solution with the desired properties for $1 < k \leq 5$ instead of restricting k to $1 < k < 4$.

However, our method of proof does not show that $\lim_{t \to 0} t^{-1}u(t) < \infty$ for $4 < k < 5$, although the existence of a continuous solution is shown for $4 < k < 5$.

3. Construction of Comparison Functions. Inequality (7) shows that $J(y)$ is bounded below because of the normalization condition $\int_0^\infty a(t)t^{1-k}y^{k+1}dt = 1$. Thus, $\inf_{A} J(y) \geq (1/25M^2)^{k+1} > 0$.

Let $\lambda = \text{g.}\ell\text{.b. } J(y)$, $(\lambda > 0)$. Then there exists a sequence of functions $\{y_n(t)\}_1^\infty$ from A, with

$$\lim_{n \to \infty} J(y_n) = \lambda. \tag{8}$$

We shall show that such a sequence $\{y_n\}$ must be uniformly bounded, and equicontinuous on $[0,\infty)$. Therefore, by Ascoli's Theorem, there exists a subsequence $\{y_{n_k}\}$ which converges to a continuous function $y(t)$, uniformly on each finite interval $[0,T]$.

Since $\lim_{n \to \infty} J(y_n) = \lambda$, the values of $J(y_n)$ are bounded, that is, there exists a positive constant C such that

$$J(y_n) \leq C^2, \quad n = 1,2,\cdots.$$

We have $y_n^2 \leq J(y_n)$. So $y_n^2(t) \leq C^2$, $\forall\, t \geq 0$. Therefore, the sequence $\{y_n\}$ is uniformly bounded on $[0,\infty)$. Now, for $0 \leq t_1 < t_2 < \infty$,

$$|y_n(t_2)-y_n(t_1)|^2 = \left(\int_{t_1}^{t_2} \dot{y}_n dt\right)^2 \leq \left(\int_{t_1}^{t_2} 1^2 dt\right)\left(\int_{t_1}^{t_2} \dot{y}_n^2 dt\right) \quad \text{(Hölder's Inequality)}$$

$$= (t_2-t_1)\int_{t_1}^{t_2} \dot{y}_n^2 dt$$

$$\leq (t_2-t_1)\int_{t_1}^{t_2} (\dot{y}_n^2+y_n^2)dt$$

$$\leq (t_2-t_1)J(y_n) \leq (t_2-t_1)C^2 .$$

Hence, $\{y_n\}$ is equicontinuous on $[0,\infty)$.

Therefore, there exists a subsequence, denoted again by $\{y_n\}$, which converges uniformly on each $[0,T]$, to a continuous function $y(t)$ and $\lim_{n\to\infty} J(y_n) = \lambda$. We cannot assert that $J(y) = \lambda$, in fact, $J(y)$ may not exist. We wish to show that $\beta y(t)$, where β is a suitable positive constant, is a solution of $\ddot{u} - u + a(t)t^{1-k}u^k = 0$ and $y(t) \to 0$ for $t \to 0$ and $t \to \infty$. Moreover, we shall show that $\lim_{t \downarrow 0} t^{-1}y(t) = A$ where A is a positive real number.

First we show that with a suitable choice of the Lagrange multipliers $\alpha_n > 0$ we can define admissible functions $u_n(t)$ as solutions to the linear boundary value problem

$$\ddot{u}_n - u_n + \alpha_n a(t)t^{1-k}y_n^k = 0, \quad u_n(0) = u_n(\infty) = 0, \quad n = 1,2,\cdots, \tag{9}$$

in such a way that $J(u_n) \leq J(y_n)$. Thus, $J(u_n) \to \lambda$. If u_n solves (9), then we can write it as

$$u_n(t) = \alpha_n \int_0^\infty g(t,\tau)a(\tau)\tau^{1-k}y_n^k(\tau)d\tau, \tag{10}$$

where $g(t,\tau)$ is the Green's function of the differential operator $L(u) \equiv \ddot{u} - u$ with boundary conditions from (9), that is

$$g(t,\tau) = \begin{cases} e^{-t} \sinh \tau, & 0 \leq \tau \leq t, \\ e^{-\tau} \sinh t, & \tau \geq t. \end{cases}$$

(9) and (10) are not equivalent since even though (10) satisfies the differential equation, it may not satisfy the boundary conditions. We shall show that (10) is indeed a solution of (9) under the conditions imposed on $y_n(t)$ and k. Let us define

$$v_n(t) = \int_0^t e^{-t}\sinh\tau\, a(\tau)\tau^{1-k}y_n^k(\tau)d\tau + \int_t^\infty e^{-\tau}\sinh t\, a(\tau)\tau^{1-k}y_n^k(\tau)d\tau$$

$$= e^{-t}\int_0^t \sinh\tau\, a(\tau)\tau^{1-k}y_n^k(\tau)d\tau + \sinh t\int_t^\infty e^{-\tau}a(\tau)\tau^{1-k}y_n^k(\tau)d\tau,$$

i.e., $u_n(t)$ when $\alpha_n = 1$. We denote the first integral on the right side by $\phi(t)$ and the second by $\psi(t)$, and we shall study the behaviour of $\phi(t)$ and $\psi(t)$ near 0, and as $t \to +\infty$.

$$\phi(t) = \int_0^t \sinh\tau\ a(\tau)\tau^{1-k}y_n^k(\tau)d\tau$$

$$= \int_0^t \frac{\sinh\tau}{\tau}\ a(\tau)\tau^{1-k}y_n^2 y_n^{k-2}\tau^{1-\frac{k}{2}}\tau^{\frac{k}{2}}\,d\tau.$$

Since $\sinh\tau \le \tau\cosh\tau$, $0 < a(t) < M$, and $y_n^2(t) \le t\,J(y_n) \le tC^2$, we have $t^{\frac{2-k}{2}}y_n^{k-2} \le C^{k-2}$, so

$$\phi(t) \le C^{k-2}M\cosh t\int_0^t \tau^{1-\frac{k}{2}}y_n^2 d\tau$$

$$\le MC^{k-2}t^{3-\frac{k}{2}}\cosh t\int_0^t \frac{y_n^2}{\tau^2}d\tau \quad \text{(since for } \tau \le t \text{ and } 1 < k \le 6, \text{ we have } \tau^{3-\frac{k}{2}} \le t^{3-\frac{k}{2}})$$

$$\le 4MC^{k-2}t^{3-\frac{k}{2}}\cosh t\int_0^t \dot{y}_n^2\,d\tau$$

$$\le 4C^kMt^{3-\frac{k}{2}}\cosh t \quad \left(\text{since } \int_0^t \frac{y_n^2}{\tau^2}d\tau \le 4\int_0^t \dot{y}_n^2 d\tau \le 4J(y_n) \le 4C^2\right).$$

Therefore,

$$\phi(t) \le 4MC^k\, t^{3-\frac{k}{2}}\cosh t \tag{11}$$

This estimate will be useful near $t = 0$. Now we study the behaviour of $\phi(t)$ near $t = \infty$.

$$\phi(t) = \int_0^t \sinh\tau\, a(\tau)\tau^{1-k} y_n^k(\tau)d\tau$$

$$= \int_0^{t_0} \sinh\tau\, a(\tau)\tau^{1-k} y_n^k(\tau)d\tau + \int_{t_0}^t \sinh\tau\, a(\tau)\tau^{1-k} y_n^k(\tau)d\tau, \quad 0 < t_0 < t$$

$$= \phi(t_0) + \int_{t_0}^t \sinh\tau\, a(\tau)\tau^{1-k} y_n^k(\tau)d\tau$$

$$\leq \phi(t_0) + \frac{1}{2} MC^k \int_{t_0}^t \tau^{1-k} e^{\tau} d\tau \quad (\text{since } y_n^k \leq C^k).$$

Set $t_0 = 2(k-1)$. Now, $\tau^{1-k} e^{\frac{\tau}{2}}$ is increasing for $\tau > 2(k-1)$, so, for $t \geq \tau > t_0$, $\tau^{1-k} e^{\frac{\tau}{2}} \leq t^{1-k} e^{\frac{t}{2}}$. Therefore,

$$\phi(t) \leq \phi(t_0) + \frac{1}{2} MC^k t^{1-k} e^{\frac{t}{2}} \int_{t_0}^t e^{\frac{t}{2}} d\tau$$

$$= \phi(t_0) + MC^k t^{1-k} e^t - MC^k t^{1-k} e^{\frac{t+t_0}{2}}$$

$$< \phi(t_0) + MC^k t^{1-k} e^t.$$

From (11), $\phi(t_0) \leq 4MC^k t_0^{3-\frac{k}{2}} \cosh t_0$, therefore,

$$\phi(t) \leq 4MC^k t_0^{3-\frac{k}{2}} \cosh t_0 + MC^k t^{1-k} e^t, \quad t_0 \geq 2(k-1), \quad 1 \leq k. \tag{12}$$

Now, for $\psi(t)$, we have for $k \geq 1$,

$$\psi(t) = \int_t^\infty e^{-\tau} a(\tau)\tau^{1-k} y_n^k(\tau)d\tau$$

$$\leq MC^k \int_t^\infty e^{-\tau}\tau^{1-k} d\tau \qquad (\text{since } y_n^k \leq C^k)$$

$$\leq MC^k t^{1-k} \int_t^\infty e^{-\tau} d\tau$$

$$= MC^k t^{1-k} e^{-t}.$$

Hence,

$$\psi(t) \le MC^k t^{1-k} e^{-t}, \quad 1 \le k, \quad t \ge 0 . \tag{13}$$

This estimate will be useful for t large. To estimate $\psi(t)$ near $t = 0$, we choose a value t in $(0,1)$ and write

$$\psi(t) = \int_t^1 \tau^{1-k} e^{-\tau} y_n^k(\tau) a(\tau) d\tau + \psi(1) .$$

We have from (13), $\psi(1) \le C^k M e^{-1} < C^k M$. Therefore,

$$\psi(t) < MC^k + \int_t^1 \tau^{1-k} e^{-\tau} y_n^k(\tau) a(\tau) d\tau$$

$$\le MC^k + M \int_t^1 \tau^{1-k} e^{-\tau} y_n^2 \, \tau^{-1+\frac{k}{2}} \frac{y_n^{k-2}}{\tau^{-1+\frac{k}{2}}} d\tau$$

$$\le MC^k + MC^{k-2} \int_t^1 y_n^2 \, \tau^{-\frac{k}{2}} e^{-\tau} d\tau \quad \left(\text{since } y_n^2 \le t\, J(y_n) \le t\, C^2 \text{ implies } \frac{y_n^{k-2}}{t^{\frac{1}{2}(k-2)}} \le C^{k-2}\right)$$

$$\le MC^k + MC^{k-2} \int_t^1 \frac{y_n^2}{\tau^2} \tau^{2-\frac{k}{2}} d\tau \quad (\text{since } e^{-\tau} \le 1) \tag{14}$$

$$\le MC^k + MC^{k-2} \int_t^1 \frac{y_n^2}{\tau^2} d\tau \quad (\text{since } \tau^{2-\frac{k}{2}} \le 1 \text{ on } [0,1] \text{ for } k \le 4)$$

$$\le MC^k + 4MC^{k-2} \int_t^1 \dot{y}^2 d\tau \quad \left(\text{since } \int_0^T \frac{y^2}{\tau^2} d\tau \le 4 \int_0^T \dot{y}^2 d\tau\right)$$

$$\le MC^k + 4MC^k \quad \left(\text{since } \int_t^1 \dot{y}^2 \le J(y) \le C^2\right)$$

$$= 5MC^k \quad (1 \le k \le 4).$$

Hence,

$$\psi(t) \le 5MC^k \text{ if } k \le 4, \quad 0 \le t \le 1. \tag{15}$$

If $k > 4$, then $\tau^{2-\frac{k}{2}} \leq t^{2-\frac{k}{2}}$ since $1 \geq \tau \geq t \geq 0$. So from (14), we have

$$\psi(t) \leq MC^k + MC^{k-2}\, t^{2-\frac{k}{2}} \int_t^1 \frac{y_n^2}{\tau^2}\, d\tau$$

$$\leq MC^k + 4MC^k\, t^{2-\frac{k}{2}} \quad (k \geq 4)$$

Therefore,

$$\psi(t) \leq MC^k\left(1+4t^{2-\frac{k}{2}}\right) \quad \text{for} \quad k>4, \quad 0 \leq t \leq 1 \tag{16}$$

Now, we shall show that the function $v_n(t) = e^{-t}\phi(t) + \sinh t\ \psi(t)$ tends to zero for both $t \to 0$ and $t \to \infty$. Using the above estimates for $\phi(t)$ and $\psi(t)$, we first show that $v_n(t) \to 0$ as $t \to 0$. For $1 \leq k \leq 4$, $0 < t \leq 1$, using (11) and (13), we obtain

$$v_n(t) \leq 4MC^k e^{-t}\, t^{3-\frac{k}{2}} \cosh t + 5MC^k \sinh t \tag{17}$$

and for $6 \geq k > 4$, $0 < t \leq 1$, we use (11) and (16) to obtain

$$v_n(t) \leq 4MC^k e^{-t}\, t^{3-\frac{k}{2}} \cosh t + MC^k\left(1+4t^{2-\frac{k}{2}}\right) \sinh t.$$

Thus, we see that $v_n(t) \to 0$ *as* $t \to 0$ *if* $1 < k < 6$.

We now will show that $v_n(t) \to 0$ as $t \to \infty$, for $1 < k$. By (12) and (13),

$$v_n(t) \leq 4MC^k\, t_0^{3-\frac{k}{2}} \cosh t_0\, e^{-t} + (1+\sinh t\ e^{-t})MC^k t^{1-k}, \quad k > 1.$$

This implies that $v_n(t) \to 0$ as $t \to \infty$ for all $k > 1$.

To estimate $\dot{v}_n(t)$ we write

$$v_n(t) = e^{-t}\int_0^t \tau^{1-k} a(\tau)\sinh\tau\, y_n^k(\tau)d\tau + \sinh t\int_t^\infty \tau^{1-k} a(\tau)e^{-\tau} y_n^k(\tau)d\tau$$

$$= e^{-t}\int_0^t \tau^{1-k} a(\tau)\sinh\tau\, y_n^k(\tau)d\tau + \sinh t\int_t^T \tau^{1-k} a(\tau)e^{-\tau} y_n^k(\tau)d\tau +$$

$$+ \sinh t\int_T^\infty \tau^{1-k} a(\tau)e^{-\tau} y_n^k(\tau)d\tau,$$

$$\dot v_n(t) = e^{-t}a(t)t^{1-k}\sinh t\, y_n^k(t) - e^{-t}\int_0^t a(\tau)\tau^{1-k}\sinh\tau\, y_n^k(\tau)d\tau$$

$$+ \cosh t\int_t^T \tau^{1-k}a(\tau)e^{-\tau}y_n^k(\tau)d\tau - (\sinh t)\, t^{1-k}a(t)e^{-t}y_n^k(t) +$$

$$+ \cosh t\int_T^\infty \tau^{1-k}a(\tau)e^{-\tau}y_n^k(\tau)d\tau$$

$$= e^{-t}a(t)t^{1-k}\sinh t\, y_n^k(t) - e^{-t}\int_0^t a(\tau)\tau^{1-k}\sinh\tau\, y_n^k(\tau)d\tau$$

$$+ \cosh t\int_t^\infty \tau^{1-k}a(\tau)e^{-\tau}y_n^k(\tau)d\tau - (\sinh t)\, t^{1-k}a(t)e^{-t}y_n^k(t).$$

So,

$$\dot v_n(t) = -e^{-t}\phi(t) + \cosh t\ \psi(t), \tag{18}$$

and using the estimates (12) and (13), we find that for $t \to +\infty$,

$$|\dot v_n(t)| \le e^{-t}\left(4MC^k t_0^{3-\frac{k}{2}}\cosh t_0 + MC^k t^{1-k}e^t\right) + \cosh t\, MC^k t^{1-k}e^{-t}$$

$$= 4MC^k\, t_0^{3-\frac{k}{2}}\cosh t_0\, e^{-t} + (e^{-t}\cosh t - 1)\, MC^k t^{1-k},$$

and *therefore we see that* $\dot v_n(t) \to 0$ *as* $t \to \infty$ *for* $k > 1$. As $t \to 0$, if $1 \le k \le 4$, from (11) and (15)

$$|\dot v_n(t)| \le 4MC^k e^{-t}\ t^{3-\frac{k}{2}}\cosh t + 5MC^k\cosh t\,.$$

Hence, $|\dot{v}_n(t)|$ is bounded near $t = 0$ if $1 \leq k \leq 4$, and since $v_n(t) \to 0$ as $t \to 0$, we obtain $\lim\limits_{t \to 0} \dot{v}_n(t)v_n(t) = 0$ for $1 < k \leq 4$. If $k > 4$, from (11) and (16), we get

$$\dot{v}_n(t) \leq -4MC^k e^{-t} t^{3-\frac{k}{2}} \cosh t + MC^k\left(1 + 4t^{2-\frac{k}{2}}\right) \cosh t$$

and

$$\dot{v}_n(t) = O\left(t^{2-\frac{k}{2}}\right) \leq N t^{2-\frac{k}{2}}$$

as $t \to 0$. Thus,

$$\dot{v}_n(t)v_n(t) \leq 4NMC^k e^{-t} t^{5-k} \cosh t + NMC^k t^{2-\frac{k}{2}} \sinh t + 4NMC^k t^{4-k} \sinh t.$$

Now, for $4 < k < 5$, $\lim\limits_{t \to 0} e^{-t} t^{5-k} \cosh t = 0$, also

$$\lim_{t \to 0} t^{2-\frac{k}{2}} \sinh t = \lim_{t \to 0} \frac{\sinh t}{t^{\frac{k}{2}-2}} = \lim_{t \to 0} \frac{\cosh t}{\left(\frac{k}{2} - 2\right) t^{\frac{k}{2}-3}}$$

by L'Hospital's Rule, and we see that $\lim\limits_{t \to 0} t^{2-\frac{k}{2}} \sinh t = 0$, $4 < k < 5$. Similarly, $\lim\limits_{t \to 0} t^{4-k} \sinh t = 0$, $4 < k < 5$. Hence, we obtain $\lim\limits_{t \to 0} \dot{v}_n(t)v_n(t) = 0$ if $1 < k < 5$. From (18) we get,

$$\ddot{v}_n(t) = -e^{-t}(\sinh t)a(t)t^{1-k}y_n^k(t) + e^{-t}\phi(t) - (\cosh t)e^{-t}a(t)t^{1-k}y_n^k(t) + \psi(t)\sinh t.$$

Therefore, $\ddot{v}_n(t)$ exists and we conclude that $v_n(t)$ is a solution of the boundary value problem

$$\ddot{v}_n - v_n + t^{1-k}a(t)y_n^k = 0, \quad v_n(0) = v_n(\infty) = 0.$$

Thus $u_n(t) = \alpha_n v_n(t)$ solves the boundary value problem (9). Also, since we assumed $y_n(t)$ to be non-negative in $(0,\infty)$, formula (10) implies $v_n(t)$ is also non-negative in $(0,\infty)$.

4. We now prove $J(u_n) \le J(y_n)$.

We have $\ddot{v}_n - v_n + a(t)t^{1-k}y_n^k = 0$, $v_n(0) = v_n(\infty) = 0$. We multiply this equation by $v_n(t)$ and integrate from 0 to T. Then,

$$\int_0^T \left(v_n\ddot{v}_n - v_n^2 + a(t)t^{1-k}y_n^k v_n\right)dt = 0.$$

Now,

$$\int_0^T v_n\ddot{v}_n\,dt = v_n\dot{v}_n\Big|_0^T - \int_0^T \dot{v}_n^2\,dt = v_n(T)\dot{v}_n(T) - v_n(0)\dot{v}_n(0) - \int_0^T \dot{v}_n^2\,dt.$$

Hence,

$$-\int_0^T \dot{v}_n^2\,dt - \int_0^T v_n^2\,dt = -\int_0^T a(t)t^{1-k}y_n^k v_n\,dt - v_n(T)\dot{v}_n(T),$$

so

$$H^2(T) \equiv \int_0^T (\dot{v}_n^2 + v_n^2)dt = \int_0^T a(t)t^{1-k}y_n^k v_n\,dt + v_n(T)\dot{v}_n(T). \tag{19}$$

Also, by Hölder's Inequality we have

$$\begin{aligned}\int_0^T a(t)t^{1-k}y_n^k v_n\,dt &= \int_0^T \left(a(t)^{\frac{k}{k+1}}\, t^{(1-k)\frac{k}{k+1}}\, y_n^k\right)\left(a(t)^{\frac{1}{k+1}}\, t^{(1-k)\frac{1}{k+1}}\, v_n\right)dt \\ &\le \left[\int_0^T a(t)t^{1-k}y_n^{k+1}dt\right]^{\frac{k}{k+1}}\left[\int_0^T a(t)t^{1-k}v_n^{k+1}dt\right]^{\frac{1}{k+1}} \qquad (20) \\ &\le \left[\int_0^T a(t)t^{1-k}v_n^{k+1}dt\right]^{\frac{1}{k+1}} \quad \left(\text{since } \int_0^\infty a(t)t^{1-k}y_n^{k+1}dt = 1\right) \\ &= \left[\left(\int_0^T a(t)t^{1-k}v_n^{k+1}dt\right)^2\right]^{\frac{1}{2(k+1)}}.\end{aligned}$$

From (7), it follows that

$$\begin{aligned}\int_0^T a(t)t^{1-k}y_n^k v_n\,dt &\le \left[25M^2\left\{\int_0^T (\dot{v}_n^2 + v_n^2)dt\right\}^{k+1}\right]^{\frac{1}{2(k+1)}} \\ &\le (5M)^{\frac{1}{k+1}} H(T) \le 5M\, H(T) \quad (\text{without loss of generality, } 5M \ge 1)\end{aligned}$$

Hence,

$$\int_0^T a(t)t^{1-k}y_n^k v_n\,dt \le 5MH(T) \tag{21}$$

From (19), we have $H^2(T) - \int_0^T a(t)t^{1-k}y_n^k v_n\,dt = v_n(T)\dot{v}_n(T)$.

Consequently, (21) implies that $H^2(T) - 5MH(T) \le \dot{v}_n(T)v_n(T)$ which in turn implies that $\left(H(T) - \frac{5}{2}M\right)^2 \le \frac{25}{4}M^2 + \dot{v}_n(T)v_n(T)$. Now, we have $\dot{v}_n(T)v_n(T) \to 0$ as $T \to \infty$. Hence

$$\left(J(v_n) - \frac{5}{2}M\right)^2 \le \frac{25}{4}M^2.$$

Thus $\int_0^\infty (\dot{v}_n^2 + v_n^2)dt$ exists and since $u_n = \alpha_n v_n$, we conclude that $\int_0^\infty (\dot{u}_n^2 + u_n^2)dt$, and $\int_0^\infty a(t)t^{1-k}u_n^{k+1}dt$, exist. (Note that $u_n(t) = \alpha_n v_n(t)$ is of constant sign, since $v_n(t)$ is non-negative). Thus, we may choose the positive normalization constant α_n such that $\int_0^\infty a(t)t^{1-k}u_n^{k+1}dt = 1$, i.e.,

$$\alpha_n = \left[\int_0^\infty a(t)t^{1-k}v_n^{k+1}dt\right]^{-\frac{1}{k+1}}.$$

It follows, then, that

$$\int_0^T a(t)t^{1-k}y_n^k u_n\,dt \le \left(\int_0^T a(t)t^{1-k}y_n^{k+1}dt\right)^{\frac{k}{k+1}}\left(\int_0^T a(t)t^{1-k}u_n^{k+1}dt\right)^{\frac{1}{k+1}} \le 1\cdot 1 = 1.$$

Therefore, from (19) (multiplied by α_n^2), we get, for $T \to \infty$,

$$\int_0^\infty (\dot{u}_n^2 + u_n^2)dt = \alpha_n\int_0^\infty a(t)t^{1-k}y_n^k u_n\,dt \le \alpha_n,$$

that is,

$$J(u_n) \equiv \int_0^\infty (\dot{u}_n^2 + u_n^2)dt \le \alpha_n\ . \tag{22}$$

Next, we multiply $\ddot{u}_n - u_n + \alpha_n a(t)t^{1-k}y_n^k = 0$ by $y_n(t)$ and integrate from 0 to ∞,

$$\int_0^\infty \left(y_n\ddot{u}_n - u_n y_n + \alpha_n a(t)t^{1-k}y_n^{k+1}\right)dt = 0. \tag{23}$$

Before we evaluate the integral (23) we show that $\dot{u}_n(t)y_n(t) \to 0$ for $t \to 0$ and $t \to \infty$. Recall that $u_n(t) = \alpha_n v_n(t)$, hence $\dot{u}_n(t) = \alpha_n\dot{v}_n(t)$. Now note that $y_n(t) = O(t^{\frac{1}{2}})$, $\dot{v}_n(t)$ is bounded near $t = 0$ if $1 < k \le 4$, and $\dot{v}_n(t) = O(t^{2-\frac{k}{2}})$ if $k > 4$. It follows then that $\dot{u}_n(t)y_n(t) = O(t^{\frac{1}{2}})$ if $1 < k \le 4$, and $\dot{u}_n(t)y_n(t) = O(t^{\frac{5-k}{2}})$ if $4 < k < 5$. This implies that $\lim_{t\to 0} \dot{u}_n(t)y_n(t) = 0$. We have shown that $\dot{v}_n(t) \to 0$ as $t \to \infty$ for $k > 1$, hence $\dot{u}_n(t) \to 0$ as $t \to \infty$; also $y_n^2(t) \le c^2$, $t \ge 0$. We thus have $\lim_{t\to\infty} \dot{u}_n(t)y_n(t) = 0$. We then obtain

$$\int_0^\infty y_n\ddot{u}_n dt = y_n(\infty)\dot{u}_n(\infty) - y_n(0)\dot{u}_n(0) - \int_0^\infty \dot{u}_n\dot{y}_n dt$$

$$= -\int_0^\infty \dot{u}_n\dot{y}_n dt.$$

So, (23) becomes

$$\int_0^\infty (\dot{u}_n\dot{y}_n + u_n y_n)dt = \alpha_n\int_0^\infty a(t)t^{1-k}y_n^{k+1}dt = \alpha_n.$$

This implies that $\int_0^\infty (\dot{u}_n^2 + u_n^2)dt + \int_0^\infty (\dot{y}_n^2 + y_n^2)dt - \int_0^\infty [(\dot{u}_n - \dot{y}_n)^2 + (u_n - y_n)^2]dt = 2\alpha_n$. Therefore,

$$J(u_n) + J(y_n) - J(u_n - y_n) = 2\alpha_n. \tag{24}$$

We have $J(u_n) \le \alpha_n$ by (22). Therefore, $J(u_n) + J(y_n) - J(u_n - y_n) \ge 2J(u_n)$ and it follows that

$$J(u_n) \le J(y_n) - J(u_n - y_n) \le J(y_n), \tag{25}$$

with equality if and only if $J(u_n - y_n) = 0$, i.e., if and only if $u_n \equiv y_n$.

Now, we proceed to find an estimate for α_n. From (22) and (24), we get $J(y_n) - J(u_n - y_n) \geq \alpha_n$, that is, $J(y_n) \geq \alpha_n + J(u_n - y_n) \geq \alpha_n \geq J(u_n)$. Therefore,

$$J(y_n) \geq \alpha_n \geq J(u_n). \tag{26}$$

Since the u_n's are admissible, $\lambda = \inf_A J(y)$, and $J(y_n)$ tends to λ, (26) implies

$$\lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} J(y_n) = \lambda \tag{27}$$

which in turn implies $\lim_{n\to\infty} J(u_n - y_n) = 0$ by (25). So, setting $J(u_n - y_n) = \varepsilon_n^2$ and noting that $y^2(t) \leq J(y)$, we get $(u_n - y_n)^2 \leq J(u_n - y_n) = \varepsilon_n^2$, that is, $(u_n - y_n)^2 \leq \varepsilon_n^2$. Thus $|u_n(t) - y_n(t)| \leq \varepsilon_n$, $\lim_{n \to \infty} \varepsilon_n = 0$.

Now,

$$u_n(t) = \alpha_n \int_0^T \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau + \alpha_n \int_T^\infty a(\tau) \tau^{1-k} g(t,\tau) y_n^k d\tau.$$

Therefore,

$$\left| y_n(t) - \alpha_n \int_0^T \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau \right| = \left| y_n(t) - u_n(t) + \alpha_n \int_T^\infty \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau \right|$$

$$\leq |y_n(t) - u_n(t)| + \left|\alpha_n \int_T^\infty \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau\right|$$

$$\leq \varepsilon_n + \left|\alpha_n \int_T^\infty \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau\right| .$$

Now,

$$\int_T^\infty \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau = \sinh t \int_T^\infty \tau^{1-k} a(\tau) e^{-\tau} y_n^k d\tau \qquad (t \leq T)$$

$$\leq (\sinh T)\, MC^k\, T^{1-k}\, e^{-T} \qquad (\text{by } (13))$$

$$\leq \frac{M}{2} C^k\, T^{1-k} \leq MC^k\, T^{1-k}.$$

Hence,

$$|y_n(t) - \alpha_n \int_0^T \tau^{1-k} a(\tau) g(t,\tau) y_n^k d\tau| \leq \varepsilon_n + \alpha_n MC^k T^{1-k} \qquad (t \leq T)$$

If $n \to \infty$, then $y_n(t) \to y(t)$ uniformly in $[0,T]$; and since $\varepsilon_n \to 0$ and $\alpha_n \to \lambda$ by (27), we get

$$|y(t) - \lambda \int_0^T \tau^{1-k} a(\tau) g(t,\tau) y^k d\tau| \leq \lambda MC^k T^{1-k}$$

which implies

$$y(t) = \lambda \int_0^\infty \tau^{1-k} a(\tau) g(t,\tau) y^k d\tau \qquad (\text{for } k > 1) . \tag{28}$$

Consequently, $y(t) = \lim_{n \to \infty} y_n(t) = \lim_{n \to \infty} u_n(t)$, $y(t)$ is a solution of the integral equation (28), and $y(t) \to 0$ as $t \to 0$ and $t \to \infty$ from similar estimates to those used on $u_n(t)$

Finally, we will show that $y(t)$ is a *non-trivial* solution of (28). Multiplying the equation $\ddot{u}_n - u_n + \alpha_n t^{1-k} a(t) y_n^k = 0$ by $u_n(t)$ and integrating from T to ∞ $(0 < T < \infty)$, we get

$$\int_T^\infty \left(u_n \ddot{u}_n - u_n^2 + \alpha_n t^{1-k} a(t) y_n^k u_n\right) dt = 0$$

i.e., $$\int_T^\infty (\dot{u}_n^2 + u_n^2) dt = \alpha_n \int_T^\infty a(t) t^{1-k} y_n^k u_n \, dt - u_n(T)\dot{u}_n(T).$$ Now

$$\int_T^\infty (\dot{u}_n^2 + u_n^2) dt = \int_0^\infty (\dot{u}_n^2 + u_n^2) dt - \int_0^T (\dot{u}_n^2 + u_n^2) dt$$

$$= J(u_n) - \int_0^T (\dot{u}_n^2 + u_n^2) dt \geq \lambda - \int_0^T (\dot{u}_n^2 + u_n^2) dt$$

and

$$\int_T^\infty t^{1-k} a(t) y_n^k u_n \, dt \leq [J(y_n)]^{\frac{k-1}{2}} M \, T^{1-k} \int_T^\infty u_n y_n \, dt$$

$$\leq [J(y_n)]^{\frac{k-1}{2}} M \, T^{1-k} [J(u_n)]^{\frac{1}{2}} [J(y_n)]^{\frac{1}{2}}$$

since for $t \geq T$ $t^{1-k} \leq T^{1-k}$ and $y_n^2(t) \leq J(y_n)$. Therefore, using the

above estimate and (19) (multiplied by α_n^2), we obtain

$$\lambda - \int^T (\dot{u}_n^2 + u_n^2)dt \leq \int_T^\infty (\dot{u}_n^2 + u_n^2)dt$$

$$= \alpha_n \int_T^\infty t^{1-k} y_n^k u_n \, a(t)dt - u_n(T)\dot{u}_n(T)$$

$$\leq \alpha_n [J(y_n)]^{\frac{k}{2}} [J(u_n)]^{\frac{1}{2}} M \, T^{1-k} - u_n(T)\dot{u}_n(T),$$

that is,

$$\lambda \leq \int_0^T (\dot{u}_n^2 + u_n^2)dt + \alpha_n [J(y_n)]^{\frac{k}{2}} [J(u_n)]^{\frac{1}{2}} M \, T^{1-k} - u_n(T)\dot{u}_n(T)$$

$$= \alpha_n \int_0^T t^{1-k} a(t) y_n^k u_n dt + u_n(T)\dot{u}_n(T) + \alpha_n [J(y_n)]^{\frac{k}{2}} [J(u_n)]^{\frac{1}{2}} M T^{1-k} - u_n(T)\dot{u}_n(T).$$

Since $J(u_n) \leq J(y_n)$,

$$\lambda \leq \alpha_n \int_0^T t^{1-k} a(t) y_n^k u_n dt + \alpha_n M \, T^{1-k} [J(y_n)]^{\frac{k+1}{2}} .$$

Suppose $y_n(t) \to y(t) \equiv 0$ uniformly on $[0,T]$ (hence $u_n(t) \to 0$) for $n \to \infty$. Then we could conclude $\lambda \leq M \, T^{1-k} \lambda^{\frac{k+3}{2}}$ since $\alpha_n \to \lambda$ and $J(y_n) \to \lambda$ as $n \to \infty$. But this is impossible since $\lambda > 0$ and the R.H.S. can be made arbitrarily small by taking T large enough. Our function $y(t)$ is therefore a non-trivial solution of the integral equation (28).

Obviously, $y(t)$ is non-negative in $(0,\infty)$. From the equation (28) we see that $y(t)$ is positive in $(0,\infty)$ because $g(t,\tau)$ is positive except at $\tau = 0$ and $\tau = \infty$, and also by the condition imposed on $a(t)$ in (1b).

We saw that the R.H.S. of (28) can be differentiated twice with respect to t, that is,

$$\ddot{y}(t) = \lambda e^{-t}\phi(t) + \lambda \sinh t \, \psi(t) - \lambda e^{-t} a(t) t^{1-k} y^k \sinh t - \lambda t^{1-k} a(t) e^{-t} y^k \cosh t$$

$$= y(t) - \lambda \, a(t) t^{1-k} y^k [e^{-t}(\sinh t + \cosh t)] = y(t) - \lambda \, a(t) t^{1-k} y^k .$$

Hence, by (28), $y(t)$ is a solution of the differential equation

$$\ddot{y} - y + \lambda t^{1-k} a(t) y^k = 0, \quad y(0) = y(\infty) = 0. \tag{29}$$

And if we let

$$u(t) = \lambda^{\frac{1}{k-1}} y(t) \tag{30}$$

we then get

$$\ddot{u} - u + t^{1-k} a(t) u^k = 0, \quad u(0) = u(\infty) = 0.$$

Consequently, for $1 < k < 5$, the equation $\ddot{u} - u + t^{1-k} a(t) u^k = 0$ has a non-trivial solution $u(t)$ such that

(i) it is continuous for $0 \leq t < \infty$,

(ii) $u(t) \to 0$ for $t \to 0$,

(iii) $u(t) \to 0$ for $t \to \infty$.

Since equation (2) is related to (1a) by letting $u(t) = t\, x(t)$, we are assured of the existence of a solution of (1a) for $0 \leq t < \infty$ only if $\lim_{t \to 0} t^{-1} u(t)$ exists. In order to show that $\lim_{t \to 0} t^{-1} u(t)$ exists, considering the case $\alpha_n = 1$ for simplicity, we first note that, from (17),

$$v_n(t) = O\left(t^{3 - \frac{k}{2}}\right) + O(t), \quad 1 \leq k \leq 4.$$

Hence, $t^{-1} v_n(t)$ is bounded as $t \to 0+$. Now

$$t^{-1} v_n(t) = t^{-1} e^{-t} \int_0^t \tau^{1-k} a(t) y_n^k(\tau) \sinh \tau \, d\tau + t^{-1} \sinh t \int_t^\infty e^{-\tau} a(\tau) \tau^{1-k} y_n^k(\tau) d\tau.$$

Then,

$$\lim_{t \to 0} t^{-1} v_n(t) = \lim_{t \downarrow 0} t^{-1} e^{-t} \int_0^t \tau^{1-k} a(\tau) y_n^k(\tau) \sinh \tau \, d\tau + \lim_{t \downarrow 0} t^{-1} \sinh t \int_t^\infty e^{-\tau} a(\tau) \tau^{1-k} y_n^k(\tau) d\tau$$

$$= e^{-0} \lim_{\tau \downarrow 0} \tau^{1-k} a(\tau) y_n^k(\tau) \sinh(\tau) + \cosh 0 \int_0^\infty e^{-\tau} a(\tau) \tau^{1-k} y_n^k(\tau) d\tau$$

$$= O\left(\tau^{1-k}\tau^{\frac{k}{2}}\tau\right)\Big|_{\tau\downarrow 0} + \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y_n^k(\tau)d\tau \quad (\text{since } y^2(t) \le tC^2)$$

$$= O\left(\tau^{\frac{4-k}{2}}\right)\Big|_{\tau\downarrow 0} + \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y_n^k(\tau)d\tau.$$

Hence, for $k < 4$, $\lim\limits_{t\downarrow 0} t^{-1}v_n(t)$ exists. Furthermore,

$$\left|t^{-1}v_n(t) - \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y_n^k(\tau)d\tau\right| < K\tau^{\frac{4-k}{2}}, \quad 0 < \tau < t.$$

So, $\left|t^{-1}v_n(t) - \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y_n^k(\tau)d\tau\right|$ is small as $t \downarrow 0$ *independent of* n. $y(t) = \lim\limits_{n\to\infty} v_n(t)$ implies

$$\left|t^{-1}y(t) - \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y^k(\tau)d\tau\right| \le \left|t^{-1}y(t)-t^{-1}v_n(t)\right| + \left|t^{-1}v_n(t) - \right.$$
$$\left. - \int_0^\infty e^{-\tau}\tau^{1-k}a(\tau)y_n^k(\tau)d\tau\right| + \left|\int_0^\infty \tau^{1-k}e^{-\tau}a(\tau)\left(y^k(\tau)-y_n^k(\tau)\right)d\tau\right|.$$

As $t \downarrow 0$, the second term on R.H.S. is $o(1)$ *independent of* n. Hence we choose δ small enough to make the second term less than $\frac{\varepsilon}{3}$ for $0 < t < \delta$. For each fixed $t \in (0,\delta)$, we can choose $n = n(t)$ to make the first term less than $\frac{\varepsilon}{3}$ and also the third term less than $\frac{\varepsilon}{3}$. Thus,

$$\left|\frac{y(t)}{t} - \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y^k(\tau)d\tau\right| < \varepsilon \quad \text{for} \quad |t| < \delta(\varepsilon),$$

and $\lim\limits_{t\downarrow 0}\left[\frac{y(t)}{t} - \int_0^\infty e^{-\tau}a(\tau)\tau^{1-k}y^k(\tau)\right] = 0$, i.e., $\lim\limits_{t\downarrow 0} t^{-1}y(t)$ exists. Consequently, $\lim\limits_{t\downarrow 0} t^{-1}u(t)$ exists. Since $u_n(t) \to y(t)$, (17) and (30) show that there exists a continuous solution of (1a) if $1 < k < 4$. We have thus shown that a solution $u\left(x(t)\right)$ of the ordinary differential equation (1a) exists with the required properties for $1 < k < 4$. Nehari's assertion that his proof works for $k = 4$ appears to be an error.

CHAPTER III

A FURTHER APPLICATION

In this section we show how the calculus of variations approach can provide an alternative proof for the existence of a solution to a problem discussed by Luning and Perry [6].

They considered the equation

$$u'' + \lambda P(x)u^{\nu+1}(x) = 0, \quad 0 < x < 1, \quad P(x) \geq 0, \quad \nu > -2, \quad (\nu \neq 0) \tag{1}$$

with homogeneous boundary conditions

$$\alpha u(0) - \beta u'(0) = 0$$

$$\gamma u(1) + \delta u'(1) = 0.$$

In addition, we assume that $P(x) \not\equiv 0$ in $C(0,1)$. For simplicity, we shall only treat the case $u(0) = u(1) = 0$. Luning and Perry showed that if $P(x) \geq 0$, $\nu > -2$ $(\gamma \neq 0)$, then for *certain* α, β, γ, δ's, there is a positive solution to the problem. Then for $y \neq 0$, $y = \lambda^{1/\nu}u$ solves $y'' + P(x)y^{\nu+1} = 0$ with the same boundary conditions.

We begin by observing that (1) is the Euler-Lagrange equation corresponding to the problem of finding the minimum of

$$J(y) = \int_0^1 y'^2 \, dx \tag{2}$$

defined on a class

$$F = C^2 \cap \left\{ y(x) \,\middle|\, \int_0^1 P(x)y^{\nu+2}(x)dx = 1,\ y(x) > 0 \text{ on } (0,1),\ \ y(0) = y(1) = 0 \right\}.$$

Thus, if $y(x)$ is a solution of the variational problem (2), then it is

also a solution of (1) such that $y(0) = y(1) = 0$.

We shall now find a lower bound for $J(y)$. Since $y(0) = 0$ we have the following

$$y^2(x) = \left(\int_0^x y' dz\right)^2 \le x\int_0^x y'^2 dz \le xJ(y) \le J(y), \quad 0 < x < 1 \tag{3}$$

Then assuming that $\int_0^1 P(x)dx = K > 0$ ($P(x)$ is non-trivial) we arrive at

$$1 = \int_0^1 P(x)y^{\nu+2}(x)dx \le K[J(y)]^{\frac{\nu+2}{2}} \quad \text{for} \quad y \in \mathcal{F},$$

that is,

$$1 = \left(\int_0^1 P(x)y^{\nu+2}(x)dx\right)^2 \le K^2[J(y)]^{\nu+2}, \quad \nu > -2. \tag{4}$$

We have thus shown that $J(y)$ is bounded below on $\mathcal{F}$, so $\inf_{\mathcal{F}} J(y) = \mu$ exists ($\mu > 0$). Then by the definition of μ, there exists a minimizing sequence of functions $\{y_k(x)\}_1^\infty$ from $\mathcal{F}$ such that $J(y_k) \downarrow \mu$ as $k \to \infty$. Since $J(y_k) \downarrow \mu$ as $k \to \infty$, there exists a positive constant C such that $J(y_k) \le C^2$, $k = 1,2,\cdots$. Therefore, by (3), $y_k^2(x) \le C^2$, $0 < x < 1$. That is, we have shown that the sequence $\{y_k(x)\}$ is uniformly bounded on $[0,1]$. We have

$$|y_k(x_2)-y_k(x_1)|^2 = \left(\int_{x_1}^{x_2} y_k' \, dx\right)^2 \le \left(\int_{x_1}^{x_2} 1^2 \, dx\right)\left(\int_{x_1}^{x_2} y_k'^2 \, dx\right)$$

$$= (x_2-x_1)\int_{x_1}^{x_2} y_k'^2 dx \le (x_2-x_1)J(y_k) \le (x_2-x_1)C^2 \text{ for } 0 < x_1 < x_2 < 1$$

so that $\{y_k(x)\}$ is an equicontinuous family. Thus, according to Arzela's theorem, we can select a subsequence denoted again by $\{y_k(x)\}$ converging uniformly to $y(x)$ where $y(x) \in C[0,1]$, $y(0) = y(1) = 0$, $y(x) \ge 0$ and $J(y_k) \downarrow \mu$. We note that we may have $y(x) \equiv 0$, or $y(x)$ might not be in $C^1[0,1]$.

We now consider the linear boundary value problem

$$u_k''(x) = -\alpha_k P(x) y_k^{\nu+1}(x), \quad u_k(0) = u_k(1) = 0, \quad k = 1,2,\cdots. \tag{5}$$

We wish to show that by choosing $\alpha_k > 0$ so that $\int_0^1 P(x) u_k^{\nu+2}(x)dx = 1$, we can define an admissible function $u_k(x)$ as a solution to (5) satisfying $J(u_k) \leq J(y_k)$ (hence, $J(u_k) \downarrow \mu$). If we let $K(x,t)$ be the Green's function of the differential operator $Lu \equiv u''$ with boundary conditions from (5), i.e.,

$$K(x,t) = \begin{cases} t(1-x), & 0 < t \leq x \\ x(1-t), & t > x, \end{cases}$$

then the solution $u_k(x)$ of (5) can be written as

$$\begin{aligned} u_k(x) &= \alpha_k \int_0^1 K(x,t) P(t) y_k^{\nu+1}(t)dt \qquad (6) \\ &= \alpha_k (1-x) \int_0^x t\, P(t) y_k^{\nu+1}(t)dt + \alpha_k x \int_x^1 (1-t) P(t) y_k^{\nu+1}(t)dt \\ &= \alpha_k\, x \int_0^1 (1-t) P(t) y_k^{\nu+1}(t)dt + \alpha_k \int_0^x (t-x) P(t) y_k^{\nu+1}(t)dt. \end{aligned}$$

We now claim that $\lim_{k \to \infty} J(u_k) = \mu$. In order to show this we first multiply the equation (5) by $u_k(x)$ and integrate from 0 to 1, that is,

$$\int_0^1 (u_k u_k'' + \alpha_k P(x) u_k y_k^{\nu+1})dx = 0.$$

Integrating the first member by parts we get

$$\int_0^1 u_k u_k'' dx = u_k(1) u_k'(1) - u_k(0) u_k'(0) - \int_0^1 u_k'^2 dx = -\int_0^1 u_k'^2 dx .$$

Hence, using Hölder's inequality we arrive at

$$\int_0^1 u_k'^2 dx = \alpha_k \int_0^1 P(x) u_k y_k^{\nu+1} dx = \alpha_k \left[\int_0^1 \left(P(x)^{\frac{\nu+1}{\nu+2}} y_k^{\nu+1}\right) \left(P(x)^{\frac{1}{\nu+2}} u_k\right) dx \right]$$

$$\leq \alpha_k \left[\int_0^1 P(x) y_k^{\nu+2} dx \right]^{\frac{\nu+1}{\nu+2}} \left[\int_0^1 P(x) u_k^{\nu+2} dx \right]^{\frac{1}{\nu+2}} = \alpha_k .$$

We then have

$$J(u_k) \leq \alpha_k .$$

Next we multiply (5) by $y_k(x)$ and obtain

$$\int_0^1 \left(y_k(x) u_k''(x) + \alpha_k P(x) y_k^{\nu+2}(x)\right) dx = 0.$$

Now,

$$\int_0^1 y_k u_k'' dx = y_k(1) u_k'(1) - y_k(0) u_k'(0) - \int_0^1 y_k' u_k' dx.$$

So,

$$\int_0^1 y_k' u_k' dx = \alpha_k \int_0^1 P(x) y_k^{\nu+2}(x) dx = \alpha_k ,$$

and

$$2\alpha_k = 2\int_0^1 y_k' u_k' dx = \int_0^1 y_k'^2 dx + \int_0^1 u_k'^2 dx - \int_0^1 (u_k' - y_k')^2 dx$$

$$= J(y_k) + J(u_k) - J(u_k - y_k).$$

Thus,

$$2J(u_k) \leq 2\alpha_k = J(y_k) + J(u_k) - J(u_k - y_k),$$

so

$$J(u_k) \leq \alpha_k \leq J(y_k). \tag{7}$$

Since $\mu = \inf_{F} J(y)$, we conclude that $J(u_k) \downarrow \mu$ as $k \to \infty$. Then since u_k's are admissible functions, from (7) we have the following

$$\lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} J(y_k) = \mu \text{ and } \lim_{k \to \infty} J(u_k - y_k) = 0.$$

If we set $J(u_k-y_k) = \varepsilon_k^2$, from (3) we get $(u_k-y_k)^2 \leq J(u_k-y_k) = \varepsilon_k^2$, and so $|u_k(x) - y_k(x)| \leq \varepsilon_k$. Adding $y_k(x)$ to both sides of (6)

$$|y_k(x) - \alpha_k\int_0^1 K(x,t)P(t)y_k^{\nu+1}(t)dt| = |y_k(x) - u_k(x)| \leq \varepsilon_k.$$

Letting $k \to \infty$, and recalling $y_k(x) \to y(x)$, we have $u_k(x) \to y(x)$ uniformly on $[0,1]$, $\alpha_k \downarrow \mu$; thus,

$$y(x) = \mu\int_0^1 K(x,t)P(t)y^{\nu+1}(t)dt \ , \tag{8}$$

and we conclude that $y(x)$ is a solution of the integral equation (8) satisfying $y(0) = y(1) = 0$ and $y(x) = \lim_{k \to \infty} y_k(x) = \lim_{k \to \infty} u_k(x)$.

Let us now show that $y(x)$ is a *non-trivial* solution of (8). We multiply the equation (5) by $u_k(x)$ and integrate from T to 1. Now, $\int_T^1 u_k u_k'' dx = u_k(1)u_k'(1) - u_k(T)u_k'(T) - \int_T^1 u_k'^2 dx$, and we obtain

$$\int_T^1 u_k'^2 dx = \alpha_k\int_T^1 P(x)u_k(x)y_k^{\nu+1}(x)dx - u_k(T)u_k'(T).$$

Also,

$$\int_T^1 u_k'^2 dx = \int_0^1 u_k'^2 dx - \int_0^T u_k'^2 dx = J(u_k) - \int_0^T u_k'^2 dx \geq \mu - \int_0^T u_k'^2 dx \ ,$$

and using the inequality (3) we obtain

$$\alpha_k\int_T^1 P(x)y_k^{\nu+1}(x)u_k(x)dx \leq \alpha_k \|P\| \ [J(y_k)]^{\frac{\nu+1}{2}} \ [J(u_k)]^{\frac{1}{2}}(1-T)$$

where $\|P\| = \max_{[0,1]} |P(x)|$.

Then,

$$\mu - \int_0^T (u_k')^2 dx \le \alpha_k \|P\| [J(y_k)]^{\frac{\nu+1}{2}} [J(u_k)]^{\frac{1}{2}} (1-T) - u_k(T)u_k'(T),$$

$$\mu \le \int_0^T (u_k')^2 dx + \alpha_k \|P\| [J(y_k)]^{\frac{\nu+1}{2}} [J(u_k)]^{\frac{1}{2}} (1-T) - u_k(T)u_k'(T)$$

$$= u_k(T)u_k'(T) + \alpha_k \int_0^T P(x) y_k^{\nu+1}(x) u_k(x) dx + \alpha_k \|P\| [J(y_k)]^{\frac{\nu+1}{2}} [J(u_k)]^{\frac{1}{2}} (1-T) - u_k(T)u_k'(T)$$

$$\le \alpha_k \int_0^T P(x) y_k^{\nu+1}(x) u_k(x) dx + \alpha_k \|P\| [J(y_k)]^{\frac{\nu+2}{2}} (1-T) \quad (\text{since } J(u_k) \le J(y_k)).$$

Suppose $y(x) \equiv 0$. Then for $k \to \infty$, $y_k(x) \to 0$ uniformly on $[0,1]$, and since α_k and $J(y_k)$ tend to $\mu > 0$, we would have

$$0 < \mu \le \mu^{\frac{\nu+2}{2}} \|P\| (1-T),$$

so as $T \uparrow 1$ we get a contradiction. Thus we have proven that $y(x)$ is a non-trivial solution of (8).

Differentiating the equation (8) twice we see that $y(x)$ is a solution of the boundary value problem

$$y'' + \lambda P(x) y^{\nu+1} = 0, \quad y(0) = y(1) = 0, \quad \nu > -2.$$

This solution is positive since $K(x,t)$ is positive for $t > 0$.

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